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MODIFICATION OF DISTORTIONS IN THE CONICAL CARTOGRAPHIC PROJECTION OF SLOVAKIA USING LAPLACE EQUATION SOLVED BY THE FINITE ELEMENT METHOD

Andrea ÁBRAHÁMOVÁ, Margita VAJSÁBLOVÁ, Marek MACÁK

Modification of distortions in the conical cartographic projection of Slovakia using Laplace equation solved by the Finite Element Method

Abstract: The Finite Element Method (FEM) proves to be a very suitable computational method for solving the Laplace equation expressing a geodetic and other problem not only in engineering practice. Selecting an effective map projection for territory is probably the most important and challenging aspect of creating a cartographic projection in terms of the achieved scale distortions. The presented paper describes different methods to design conformal conic projections for the Slovakia territory and their optimizations based on solving the Laplace equation (LE) by the Finite Element Method (FEM). The modification was applied to four proposed cartographic projections, namely, Lambert's conformal conic projection in a normal aspect (LCC_SK), Conformal conic projection in a normal aspect with minimizing Root Mean Square (RMS) of the scale distortion values, Conformal conic projection in an oblique aspect (different than the currently used Křovák's projection) and Conformal conic projection in an oblique aspect with minimizing RMS of the scale distortion values. The results of the analysis of the proposed cartographic projections and their modifications using LE solved by FEM showed the suitability of the projections according to various criteria for the distortion of the elements of the displayed locality.

Keywords: Finite Element Method, Laplace equation, ANSYS, conformal conic projection, scale distortion, Root Mean Square

Introduction

The Finite Element Method (FEM) is one of the best-known computational methods to solve the Laplace equation for various mathematical models of physical problems that we encounter in the engineering sector under loads and Boundary Conditions (BC). Typical classes of engineering tasks that can be solved using FEM are e.g. flow simulation of heat and fluids, analysis of stress and strain curves, modelling of Earth´s field of gravity, or solving geodetic boundary value problems (Zienkiewicz, 1977).

The distortion values of the projected elements on the maps depend on the approach to creating the cartographic projection. In 2010, based on the requirements of the Geodesy, Cartography and Cadastre Authority of Slovak Republic (SR), a design of a conformal conical projection in the normal aspect was created (Vajsáblová, 2011) with the working title Lambert conformal conic projection SR (LCC_SK). Different types of conical projections of the territory of Slovakia have been published in (Vajsáblová, 2015), both in the normal aspect and in the oblique aspect. The parameters of these projections can be determined from various parallel distortion criteria, as well as from optimizing scale distortion on the land by minimizing Root Mean Square (RMS) values of length distortion on the entire land area (Vajsáblová, 2015), (Szatmári, Vajsáblová and Mojšová, 2017).

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Several authors have dealt with the application of the Laplace equation (LE) in cartography using variational methods, e. g. (Szatmári, 2015), (Hojovec, 1994), (Hojovec et al, 1996), (Skořepa and Dušek, 1997), (Bořík, 1999), (Frankich, 1982), (Szatmári, Vajsáblová and Mojšová, 2017). In the paper, we will show the modification of distortion values based on solving the Laplace equation by the Finite Element Method (FEM) with a Boundary Condition based on planar coordinates and values of scale distortion at the border of the territory of Slovakia, in four conical projections:

– Lambert´s conformal conic projection SR (LCC_SK) in a normal aspect,
– Conformal conic projection with minimizing RMS values of scale distortion in a normal aspect,
– Conformal conic projection in an oblique aspect,
– Conformal conic projection with minimizing RMS values of scale distortion in an oblique aspect.

1. Conformal conic projections of the territory Slovakia

Due to the oblong shape of the territory of Slovakia, we calculate conical projections and their modification by solving Laplace equation by the Finite Element Method. From a practical point of view, they provide comfort in the clarity of scale distortion, as isometric lines are in the image of parallel circles. The criteria for calculating the parameters of conical projections have several variants, according to the number of preserved parallels, their direct selection, or determinations from distortion of extreme parallels and basic parallel circles (Vajsáblová, 2021). Non-classical approaches include the calculation of parameters with the condition of minimizing Airy-Kavraiskii's variational criterion (integral or sum), in the area of the projected area:

$$I^2 = \frac{1}{p} \iint_{\Omega} (\ln m)^2 \, \mathrm{d}\Omega \quad \text{or} \quad I^2 = \frac{1}{n}\sum_{i=1}^{n}(m-1) \tag{1}$$

where p is the area value of the domain Ω, m is the scale distortion factor in a map projection. We will show the application of Airy-Kavraiskii's variational criterion on the creation of a conformal conic projection with the minimization of Root Mean Square of the scale distortion on the projected area.

1.1 Lambert's conformal conic projection and the Conformal conic projection with minimizing Root Mean Square of scale distortion

Conformal conic projections introduced by Johannes Heinrich Lambert (1728 – 1777) are appropriate for oblong territories along geographic parallels. The map equations of Lambert's conformal conic projection for the point with ellipsoidal coordinates φ and λ are:

$$\rho = \rho_0 \left[\frac{\tan\left(\frac{\varphi_0}{2} + 45°\right)}{\tan\left(\frac{\varphi}{2} + 45°\right)} \sqrt{\left(\frac{(1-e\sin\varphi_0)(1+e\sin\varphi)}{(1+e\sin\varphi_0)(1-e\sin\varphi)}\right)^e} \right]^n, \quad \varepsilon = n(\lambda - \lambda_0), \tag{2}$$

where λ_0 is the ellipsoidal longitude of the standard meridian, φ_0 is the ellipsoidal latitude of the standard parallel and ρ_0 is its polar radius, e is the eccentricity of the reference ellipsoid. The parameters ρ_0, n, and φ_0 are the constants of the conic projection affecting the accuracy of cartographic projection.

By modifying map equations (2) to minimize the Root Mean Square value of the scale distortion in the territory, the parameters ρ_0 and φ_0 are replaced by the parameter k by substitution:

$$k = \rho_0 \left(\tan\left(\frac{\varphi_0}{2} + 45°\right) \sqrt{\left(\frac{1-e\sin\varphi_0}{1+e\sin\varphi_0}\right)^e} \right)^n, \tag{3}$$

then the map equations of conformal conic projection with minimizing RMS are:

$$\rho = \frac{k}{\tan^n\left(\frac{\varphi}{2} + 45°\right)} \sqrt{\left(\frac{1+e\sin\varphi}{1-e\sin\varphi}\right)^{e\,n}}, \quad \varepsilon = n(\lambda - \lambda_0). \tag{4}$$

The scale distortion factor m of a conformal conic projection is calculated by:

$$m = \frac{n\rho}{N\cos\varphi}, \tag{5}$$

where N is a radius of curvature in the prime vertical:

$$N = \frac{a}{\sqrt{1-e^2\sin^2\varphi}}. \tag{6}$$

The procedure for minimizing the Root Mean Square of scale distortion of the conical projection in the projected area is based on Airy-Kavraiskii's variational criterion I (1). The projected territory is divided by ellipsoidal latitude φ to j segments (parallel bands) Δp_i with area p_i, for $i = 1, \ldots, j$:

$$I^2 = \frac{\sum_{i=1}^{j} p_i \ln^2 m_i}{\sum_{i=1}^{j} p_i}, \text{ where } \sum_{i=1}^{j} p_i = p. \tag{7}$$

The scale distortion factor m_i of the conformal conic projection of the ellipsoid for the determined i-th parallel band of the area is evaluated after the substitution ρ from (4) to the equation (5):

$$m_i = \frac{n\,k}{N_i\cos\varphi_i\tan^n\left(\frac{\varphi_i}{2}+45°\right)}\sqrt{\left(\frac{1+e\sin\varphi_i}{1-e\sin\varphi_i}\right)^{e\,n}}, \tag{8}$$

where φ_i is the ellipsoidal latitude of medial parallel of the i-th band and N_i is its radius of curvature in the prime vertical. From previous we express $\ln m_i$ which can be expressed using (8):

$$\begin{aligned} \ln m_i &= \ln(n\,k) - \ln(N_i\cos\varphi_i) + \\ &+ n\left(-\ln\tan\left(\frac{\varphi_i}{2}+45°\right) + \frac{e}{2}\ln(1+e\sin\varphi_i) - \frac{e}{2}\ln(1-e\sin\varphi_i)\right) \end{aligned} \tag{9}$$

and after the following substitutions:

$$\begin{aligned} & b = \ln(n\,k),\ \ \gamma_i = -\ln(N_i\cos\varphi_i), \\ & \alpha_i = -\ln\tan\left(\frac{\varphi_i}{2}+45°\right) + \frac{e}{2}\ln(1+e\sin\varphi_i) - \frac{e}{2}\ln(1-e\sin\varphi_i) \end{aligned} \tag{10}$$

we can evaluate the coefficients α_i and γ_i for each of the bands and formulate j equations, whereby the equation for the i-th band of the projected territory is:

$$\ln m_i = \alpha_i n + b + \gamma_i. \tag{11}$$

After substitution to the variational criterion, I in (7) is a function of two variables n and b. We obtain the minimal value of I, if the partial derivative of this function is equal to zero:

$$\begin{aligned} & n\sum_{i=1}^{j} p_i\alpha_i^2 + b\sum_{i=1}^{j} p_i\alpha_i + \sum_{i=1}^{j} p_i\alpha_i\gamma_i = 0, \\ & n\sum_{i=1}^{j} p_i\alpha_i + b\sum_{i=1}^{j} p_i + \sum_{i=1}^{j} p_i\gamma_i = 0. \end{aligned} \tag{12}$$

Therefore the normal equations are:

$$\frac{\partial\sum_{i=1}^{j} p_i(\alpha_i n + b + \gamma_i)^2}{\partial n} = 0,\ \ \frac{\partial\sum_{i=1}^{j} p_i(\alpha_i n + b + \gamma_i)^2}{\partial b} = 0. \tag{13}$$

The parameter n and the coefficient b are the solutions of this system of equations (13). Then the parameter k is evaluated from (10):

$$k = \frac{e^b}{n}. \tag{14}$$

The ellipsoidal latitudes φ_1 and φ_2 of the preserved parallels can be calculated for example by Newton's method from the conditions that their scale distortions (5) are equal to 1.

Equations for the calculation of planar rectangular coordinates x and y are:

$$\begin{aligned} x &= x_V + \rho \sin \varepsilon, \\ y &= y_V - \rho \cos \varepsilon, \end{aligned} \tag{15}$$

where x_V and y_V are the planar coordinates of the Pole (geographic or cartographic).

1.2 Parameters and scale distortions of conformal conic projections of Slovakia territory

In this paper, we present the analysis of four alternatives of conical projections of the Slovakia territory mentioned in the introduction and their modification using the Laplace equation solved by the Finite Element Method.

A) Lambert´s conformal conic projection in a normal aspect

The first of them is a cartographic projection based on the requirements of the Geodesy, Cartography and Cadaster Authority of Slovakia, which was proposed in 2010 and published in (Vajsáblová, 2011). The parameters of the Lambert conformal conic projection of the Slovak Republic were calculated by criteria of scale distortion of selected parallels. The values of parameters are:

$\lambda_0 = 19° \, 40'$,
$\varphi_0 = 48° \, 40' \, 30''$,
$n = 0.7509932274123$,
$\rho_0 = 5{,}618{,}372.319$ m.

Ellipsoidal latitudes of preserved parallels are:

$\varphi_1 = 48° \, 0' \, 30''$,
$\varphi_2 = 49° \, 20' \, 30''$.

Polar coordinates ρ and ε are calculated by (2) and rectangular coordinates by (15), where:

$x_V = 400{,}000$ m,
$y_V = 5{,}750{,}000$ m.

In the Lambert conformal conic projection of the Slovak Republic in a normal aspect, values of the scale distortion are from –6.7 to +6.7 cm/km and the RMS value of scale distortion according to Airy-Kavraiskii's variational criterion (1) equals 5.0 cm/km.

B) Conformal conic projection with minimizing Root Mean Square of scale distortion in a normal aspect

An alternative method to calculate the parameters of a conformal conic projection with the requirement of a minimal RMS value of scale distortion for the whole projected territory is described in chapter 1.1.

Within this method, the Slovak Republic onto the reference ellipsoid GRS80 between parallels with latitudes $\varphi_S = 47° \, 43' \, 09.6235''$ and $\varphi_N = 49° \, 36' \, 04.6826''$ is projected on the conical surface in a normal aspect. The parameters n and k of a conformal conic projection optimized by minimizing the RMS value of scale distortion after dividing the territory of Slovakia into 20 segments are (Szatmári, Vajsáblová and Mojšová, 2017):

$n = 0.7509555138$,
$k = 11{,}642{,}467.97$ m.

Ellipsoidal latitudes of preserved parallels are calculated by Newton's method:

$\varphi_1 = 48° \, 07' \, 45.6717''$,
$\varphi_2 = 49° \, 12' \, 54.3553''$.

Polar coordinates ρ and ε are calculated by (4) and rectangular coordinates by (15) with the same values of x_V and y_V as in projection **A)**. Then the scale distortions of this projection are from –4.4 to +9.0 cm/km and the RMS value of scale distortion in Slovakia according to Airy-Kavraiskii's variational criterion (7) is 3.4 cm/km.

C) Conformal conic projection in an oblique aspect

Another conformal conic projection effective for the territory of Slovakia is in the oblique aspect, which was described in (Vajsáblová, 2015). The cartographic pole was determined from three points on the border of Slovakia to minimize the amplitude of cartographic parallels, its spherical coordinates are:

U_K = -5° 53′ 41.1964″,
V_K = 32° 08′ 18.5219″.

Conformal conic projection in an oblique position is realized by four transformation steps. The coordinates φ and λ points on the ellipsoid GRS80 are transformed into a sphere, the spherical coordinates U and V of the points are transformed into cartographic spherical coordinates S and D, and then the polar coordinates ρ and ε in a conformal conical projection are calculated by map projection (2) with using spherical cartographic coordinates S and D. The last step is the calculation of planar rectangular coordinates x and y with using (15), where:

x_V = 1,200,000 m,
y_V = -9,200,000 m.

Values of parameters ρ_0, n, and cartographic latitude S_0 of the standard parallel are:

n = 0.5626808112205,
S_0 = 34° 14′ 29.02992″,
ρ_0 = 9,374,035.45 m.

Cartographic latitudes S_1 and S_2 of the preserved parallels are calculated by Newton's method:

S_1 = 33° 38′ 35.54602″,
S_2 = 34° 50′ 17.42458″.

Scale distortions in this conformal conic projection in an oblique aspect are from –5.4 to +5.4 cm/km and the RMS value of scale distortion in Slovakia according to Airy-Kavraiskii's variational criterion (1) is equal to 4.0 cm/km.

D) Conformal conic projection with minimizing Root Mean Square of scale distortion in an oblique aspect

The last analyzed conformal conic projection of the Slovak Republic is in an oblique aspect with RMS minimization using the Airy-Kavraiskii's variational criterion was defined in (Vajsáblová, 2015). The spherical coordinates, U_K and V_K of the cartographic pole are the same as in the previous projection and this conical projection in the oblique aspect is realized by four steps of transformation mentioned in previous projection-type **C)**, where the map projection (4) is applied with using spherical cartographic coordinates S and D.

Parameters n and k of conformal conical projection in an oblique aspect with RMS minimization have been calculated for the territory of the Slovak Republic bounded parallel band from cartographic latitude S_J = 35° 05′ 05.7509″ to S_S = 33° 23′ 42.1307″. The parameters n and k of a conformal conic projection in an oblique aspect optimized by minimizing the RMS value of scale distortion after dividing the territory of Slovakia into 20 cartographic parallel bands are:

n = 0.5626645035,
k = 13,413,558.57 m.

Cartographic latitudes S_1 and S_2 of the preserved parallels are calculated by Newton's method:

S_1 = 33° 45′ 9.29698″,
S_2 = 34° 43′ 37.2411″.

The scale distortions in this cartographic projection acquire values on the extreme cartographic parallels +7,3 cm/km and there is a scale distortion on the standard cartographic parallel -3,6 cm/km and RMS value of scale distortion according to Airy-Kavraiskii's variational criterion (1) equals to 2.7 cm/km.

2. Modification of cartographic projections using Laplace equation solved by the Finite Element Method

The use of numerical methods in mathematical cartography is a non-classical approach since we no longer look for map equations of cartographic projections as an analytical relationship between coordinates. The relationship of reference (spherical or ellipsoidal) and planar coordinates is given by the relationship of nodal points. In general, we do not find the exact solution of the differential equation, but we look for an approximate solution, e.g. we look for values that approximate the exact solution in a set of the points (Ábrahámová, 2020).

For a conformal projection Laplace equation holds:

$$\Delta u = 0 \text{ in } \Omega \tag{16}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{17}$$

and x, y are Cartesian coordinates and u is the function for the calculation of the scale distortion factor m.

The Finite Element Method was applied for the formula (16), which will be described in the next chapters.

2.1 Finite Element Method

Finite Element Method is one of the most widespread and well-known numerical methods and is used to solve various engineering problems (Reddy, 1993).

The solutions of even simple Partial Differential Equations (PDE) cannot be obtained on a uniform grid when a problem is given on complicated geometry. The FEM considers a complicated geometry as a collection of subdomains. The computational domain Ω can be approximated by a union of triangles or rectangles in 2D case, or tetrahedrons or hexahedrons in 3D case (Reddy, 1993).

In the first step of the algorithm, we discretize the original domain Ω into a collection of elements Ω^E, $E = 1, \ldots, N$. We choose the number, shape, and type of elements according to the required accuracy of the solution.

We consider the Laplace equation for a conformal projection (16). The next step is to convert the original differential form *("strong form")* of the PDE into an integral form *("weak form")*. We multiply the differential function (16) by weight function w:

$$(\Delta u) w = 0 w \tag{18}$$

and integrate it over the element Ω^E and obtain

$$\int_{\Omega^E} (\Delta u) w \, \mathrm{d}\Omega = \int_{\partial\Omega^E} 0 \, \mathrm{d}\, w \tag{19}$$

By using the Green formula, we obtain:

$$\int_{\Omega^E} \left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial w}{\partial y} \right) \mathrm{d}\Omega - \int_{\partial\Omega^E} w \left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) \mathrm{d}S = 0 \tag{20}$$

where n_x and n_y are the unit exterior normal vector's components to the boundary of Ω^E.

The second step of the algorithm is the construction of the approximation functions ψ_j^E on the element Ω^E. We will look for an approximate solution on the element in the form of a polynomial, as many times differentiable, as many times as the weak formulation requires. Derivatives that occur in a weak formulation must be non-zero. Furthermore, it must be a complete polynomial, it must contain all the exponentiation of the variable from the lowest to the highest, and at nodal points, the approximation solution must match the exact solution.

Subsequently, we adjust the approximate solution to the shape of a linear combination of functions, while the coefficients of the linear combination will be the nodal values of the unknown function. Then the approximation functions on the element Ω^E are just those functions that appear in this linear combination. Such a linear combination use is called the Galerkin decomposition and has the form:

$$u^E(x,y) = \sum_{j=1}^{N} u_j^E \psi_j^E(x,y) \tag{21}$$

where $\psi_j^E(x, y)$ are the so-called approximation functions for an element Ω^E and u_j^E are approximate values of the solution in the nodes of the element Ω^E, and N is the number of nodes of the element Ω^E (Reddy, 1993).

Substituting (21) into (20) for u and $\psi_j^E(x,y)$ for w, for all j, we obtain a system of linear equations on element Ω^E, which can be written in the compact matrix form as follows (Reddy, 1993):

$$\mathbf{K}^E_{(nxn)}\mathbf{u}^E_{(nx1)} = \mathbf{f}^E_{(nx1)}, \tag{22}$$

where $\mathbf{f}^E = f_j^E + Q_j^E$.

$\mathbf{K}^E$ is called the element stiffness matrix-vector $\mathbf{u}^E$ is the vector of unknowns and $\mathbf{f}^E$ is the right-hand side of the matrix form and Q_j^E are flows across element boundaries. (Reddy, 1993).

The last step of the FEM algorithm is assembling and then solving the global system of algebraic equations. To connect element equations to a global system are two principles, the continuity of the solution and the balance of inter-element flows used.

To get the system with a unique solution we have to apply the Dirichlet Boundary Condition for boundary nodes (Reddy, 1993).

2.2 Modification of proposed projections using Laplace equation solved by FEM

Our goal was to obtain the scale distortion m within the given domain Ω based on solving the Laplace equation (LE) for conformal projections (17), while at the boundary of the given domain Ω we prescribe the values of the scale distortion m (Ábrahámová, 2020).

The projection quality was evaluated based on Airy-Kavraiskii's variational criterion. We considered scale distortions over the discretization area. Airy-Kavraiskii's criterion takes into account the mean square scale distortion at each point of the area, calculated from extreme-scale distortions. The characteristic value of the scale distortion for the entire projected part of the reference surface is the value for n points by the sum variant of Airy-Kavraiskii's criterion (1).

A numerical solution of LE solved by FEM was applied to the territory of the Slovak Republic in types **A)**, **B)**, **C)** and **D)** of cartographic projections mentioned in the introduction. At the beginning of the numerical solution, we obtained 211 discretization points with ellipsoidal coordinates φ and λ on the boundary of the domain $\partial\Omega$. Ellipsoidal coordinates φ and λ were transformed to Cartesian coordinates x and y according to the procedure described in chapter 1.2. At the boundary, we prescribed the scale distortion m from the formula (5). The numerical solution was calculated in ANSYS 2019 R3 (ANSYS, 2019).

The numerical solution contains all values of the scale distortion factor m for all 55,517 discrete points in the case of the normal aspect and for all 65,870 discrete points in the oblique aspect, which are the result of the numerical method FEM with Cartesian coordinates x and y of the domain Ω.

In terms of normal and oblique aspects in the following chapters, a mutual comparison of alternative conformal projections in the same aspects, which were projected by the mentioned approaches **A)**, **B)**, **C)** and **D)**, will be performed.

In the analysed cartographic projection types **A)**, **B)** and their modification using LE solved by numerical method FEM, we calculated the characteristics of distortions performed in Tab. 1. In the proposal of Lambert's projection of the Slovak Republic is a requirement for the same absolute value of the scale distortion on the extreme parallels and the standard parallel, then the maximum values of scale distortion are ±6.7 cm/km. The scale distortion values of the Conformal conic projection in a normal aspect with minimizing RMS are from –4.4 cm/km to +9.0 cm/km. The below analysis showed that the obtained extremal scale distortions in these projections and cartographic projections using LE solved by FEM are almost the same, but they have a more effective distribution of the scale distortion on the projected territory, what the value of Airy-Kavraiskii's variational criterion expresses. In terms of extreme evaluation of all proposed conformal cartographic projections in the normal aspect, the mentioned Conic projection in a normal aspect, so-called Lambert´s projection has the smallest extreme values.

Airy-Kavraiskii’s variational criterion for the modification of Lambert’s projection using LE solved by FEM is 3.2 cm/km, which is approximately one and half times smaller than in the originally proposed projection, where this criterion is 5.0 cm/km. The same is true for the values of Airy-Kavraiskii’s variational criterion in the comparison of Conformal conic projection in a normal aspect with minimizing RMS (3.4 cm/km) and its modification using LE solved by FEM (2.4 cm/km). The proposed modified projections using LE solved by the numerical method FEM have a significantly more efficient distortion over a larger area.

Tab. 1 Scale distortion values of the domain Ω in the conformal conic projections in a normal aspect and their modifications using LE solved by FEM

Method of calculation	Scale distortion values		Airy-Kavraiskii’s variational criterion
	From	To	
Lambert’s projection	-6.7 cm/km	6.7 cm/km	5.0 cm/km
Modification of Lambert’s projection using LE solved by FEM	-6.7 cm/km	6.8 cm/km	3.2 cm/km
Conformal conic projection with minimizing RMS in a normal aspect	-4.4 cm/km	9.0 cm/km	3.4 cm/km
Modification of Conformal conic projection with minimizing RMS in a normal aspect using LE solved by FEM	-4.4 cm/km	9.0 cm/km	2.4 cm/km

In the following figures (Fig. 1 – 4), the *x* – coordinates are in the interval approximately from 191,000 m to 613,000 m, and the *y* – coordinates are in the interval approximately from 26,000 m to 235,000 m. Fig. 1 performs the isometric lines of scale distortion factor *m* for the Slovak Republic in the proposed Lambert’s conformal conic projection in a normal aspect. Fig. 2 shows the isometric lines of scale distortion factor m for the territory of Slovakia for the modification of Lambert’s projection using LE solved by the numerical method FEM. The isometric lines of scale distortion factor *m* in the proposal of the Conformal conic projection with minimizing RMS of scale distortions in a normal aspect are in Fig. 3 shown. Fig. 4 presents the isometric lines of scale distortion factor *m* from the modification of Conformal conic projection with minimizing RMS in a normal aspect using LE solved by FEM. The undistorted isometric lines are marked in blue in all figures.

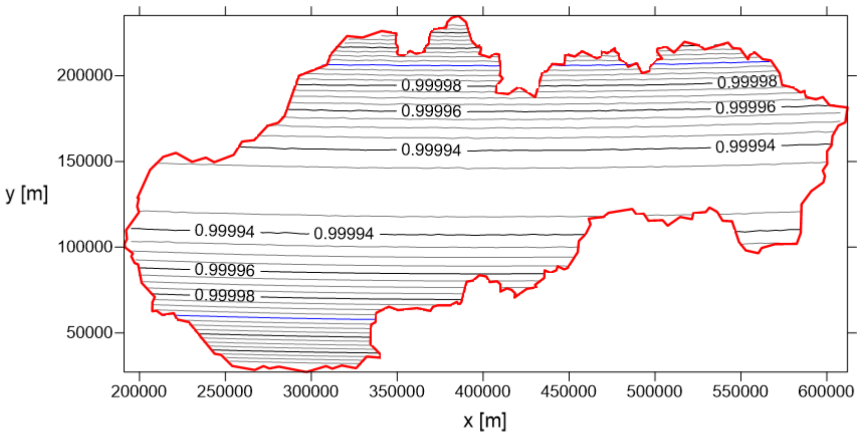


Fig. 1 Isometric lines of the scale distortion factor *m* for Lambert’s conformal conic projection in a normal aspect

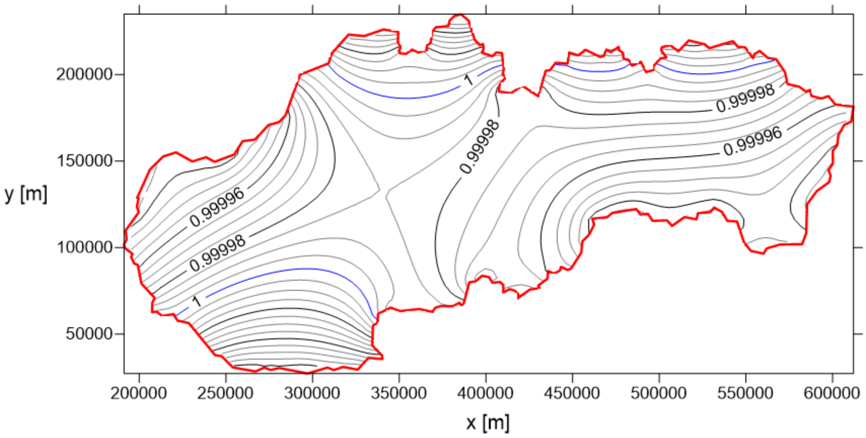


Fig. 2 Isometric lines of the scale distortion factor *m* for modification of Lambert's conformal conic projection in a normal aspect using LE solved by FEM

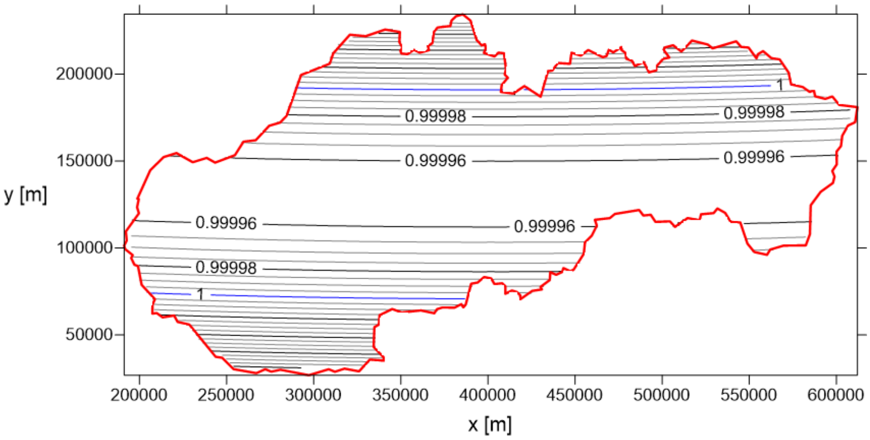


Fig. 3 Isometric lines of the scale distortion factor *m* for the Conformal conic projection with minimizing RMS in a normal aspect

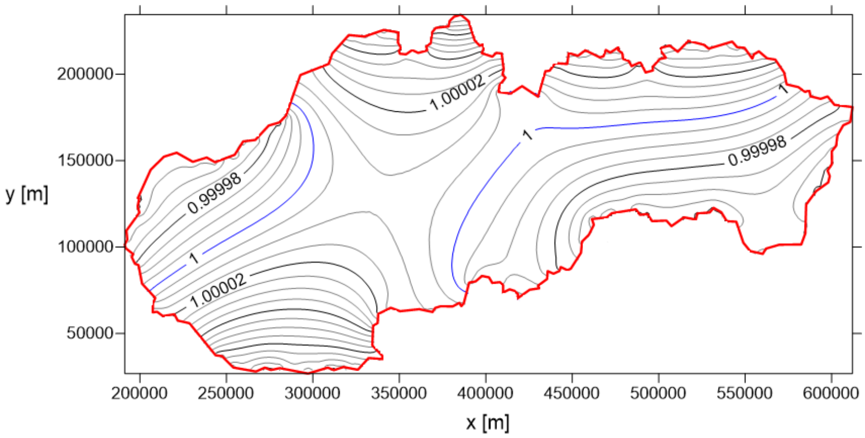


Fig. 4 Isometric lines of the scale distortion factor *m* for modification of the Conformal conic projection with minimizing RMS in a normal aspect using LE solved by FEM

The percentage representation of the scale distortion values in the Slovak Republic according to the given intervals (range 2 cm/km) in cartographic projection types **A)**, **B)** and their modification using LE solved by FEM is shown in graph (Fig. 5). The scale distortion in Lambert´s projection is expressed by a dark red color, which shows that 34% of the projected area has a scale distortion from –4 cm/km to +4 cm/km. The Conformal conic projection with minimizing RMS in a normal aspect is performed in dark green and shows that up to 94% of the area has a scale distortion value from –4 cm/km to +4 cm/km. We can see, that the optimized Conformal conic projection with RMS has smaller distortion over a larger area. In the modified projections using LE solved by the numerical method FEM, we can see that the percentage distribution of the scale distortion values is uniform, the individual columns copy the shape of the Gaussian curve, i.e. the normal probability distribution. The largest percentage values of scale distortion are concentrated around the mean value of distortions in absolute value. The modification of Conformal conic projection with minimizing RMS in a normal aspect is indicated by the light green column, which presents that approximately 92% of the territory has scale distortion values in the interval from –4 cm/km to +4 cm/km.

Based on these results we can say that the numerical method for solving LE using FEM has a better distribution of scale distortions throughout the territory.

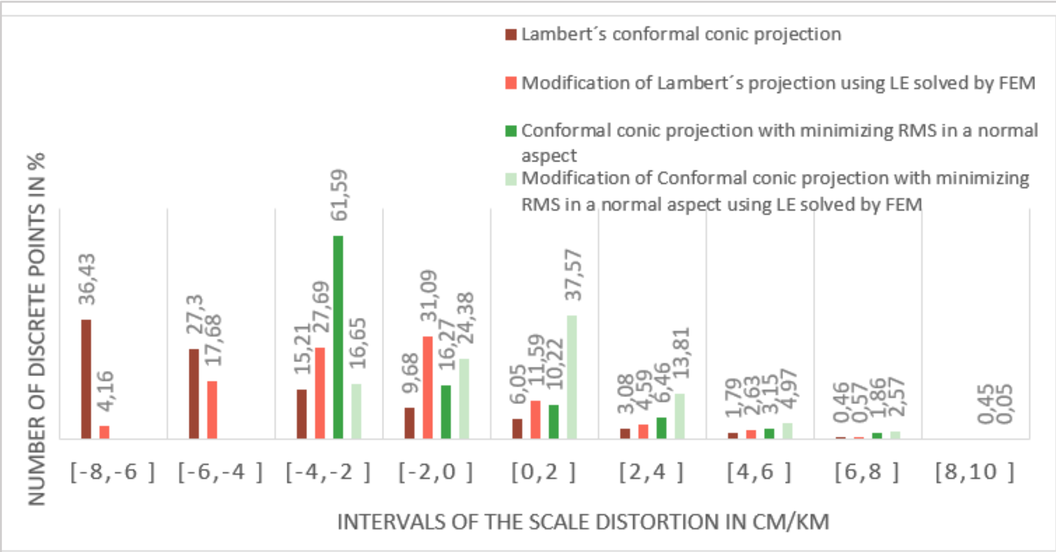


Fig. 5 The percentage of the scale distortion values in the territory of Slovakia according to the intervals in projection types **A)** and **B)** and their modifications using LE solved by FEM

In Tab. 2 are listed the scale distortion values calculated in the original proposal of the Conformal conic projection in an oblique aspect, the Conformal conic with minimizing RMS of the scale distortion values projection in an oblique aspect, and modifications of both projections using LE solved by numerical method FEM. In terms of extreme criteria, the proposal of Conformal conic projection in an oblique aspect, which is designed using the criterion of the same absolute value of scale distortion at the extreme and standard parallels, when the maximum scale distortion is ±5.4 cm/km, has the best results. The maximum values of the scale distortion in the modification of mentioned projection using LE solved by FEM are almost the same, there are from –5.4 cm/km to +5.8 cm/km. The scale distortion values in the Conformal conic projection with minimizing RMS of the scale distortion values projection in an oblique aspect are in intervals from –3.6 cm/km to +7.6 cm/km. Values of the scale distortion are similar in the modified projection using LE solved by FEM, whose maximum values are from -4 cm/km to +8 cm/km.

According to Airy-Kavraiskii's variational criterion applied to the territory of Slovakia has the best values the modification of Conformal conic projection in an oblique aspect with minimizing RMS in an oblique aspect using LE solved by FEM, namely 1.9 cm/km. The Conformal conic projection in an oblique aspect has according to Airy-Kavraiskii's variational criterion 4.0 cm/km. In comparison, this criterion is achieved in the modification of mentioned projection 2.1 cm/km, which is twice as small as in the original proposal. According to Tab. 2, it can be said that the values obtained in the modified proposals are the smallest, so they have a significantly more efficient distribution of scale distortions throughout the territory.

Tab. 2 Scale distortion values of the domain Ω in the conformal conic projections in an oblique aspect and their modifications using LE solved by FEM

Method of calculation	Scale distortion values		Airy-Kavraiskii's variational criterion
	From	To	
Conformal conic projection in an oblique aspect	-5.4 cm/km	5.4 cm/km	4.0 cm/km
Modification of Conformal conic projection in an oblique aspect using LE solved by FEM	-5.4 cm/km	5.8 cm/km	2.1 cm/km
Conformal conic projection with minimizing RMS in an oblique aspect	-3.6 cm/km	7.6 cm/km	2.7 cm/km
Modification of Conformal conic projection with minimizing RMS in an oblique aspect using LE solved by FEM	-4.0 cm/km	8.0 cm/km	1.9 cm/km

In the following figures (Fig. 6-9), the x – coordinates are in the interval approximately from 76,000 m to 505,000 m, and the y – coordinates in the interval approximately from 24,000 m to 218,000 m. Fig. 6 presents the isometric lines of scale distortion factor m for the territory of the Slovak Republic in the proposal of the Conformal conic projection in an oblique aspect. The isometric lines of the scale distortion factor m in the modification of Conformal conic projection in an oblique aspect by the numerical method for solving LE by FEM are in Fig. 7 shown. In Fig. 8, the isometric lines of scale distortion factor m in the Conformal conic projection with minimizing RMS of scale distortions in an oblique aspect are presented. Fig. 9 shows the isometric lines of scale distortion factor m for the territory of Slovakia for the modification of the mentioned projection using LE solved by the numerical method FEM. The undistorted isometric lines are marked in blue in all figures.

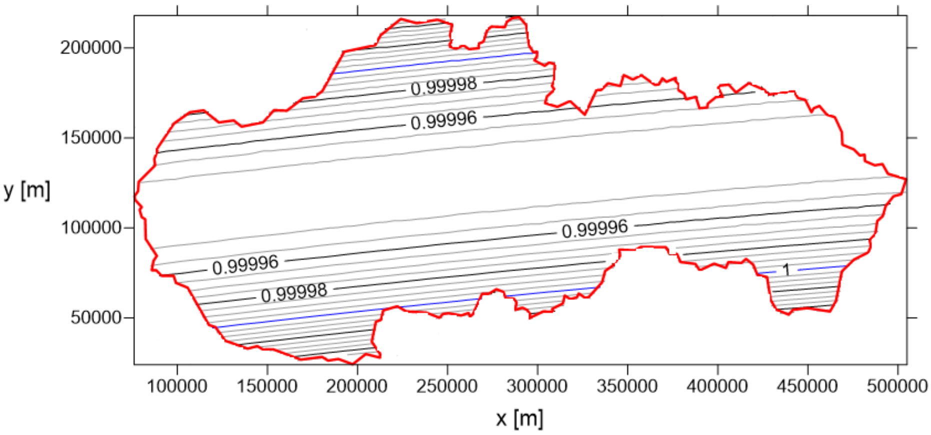


Fig. 6 Isometric lines of the scale distortion factor m for the Conformal conic projection in an oblique aspect

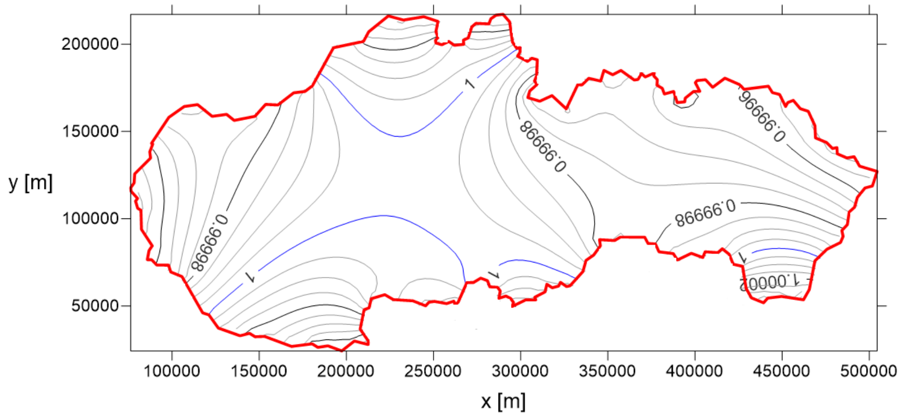


Fig. 7 Isometric lines of the scale distortion factor *m* for the Conformal conic projection in an oblique aspect using LE solved by FEM

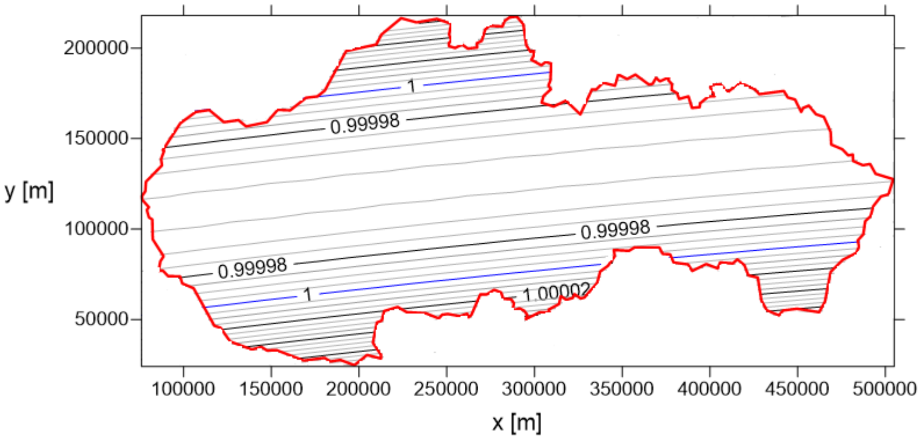
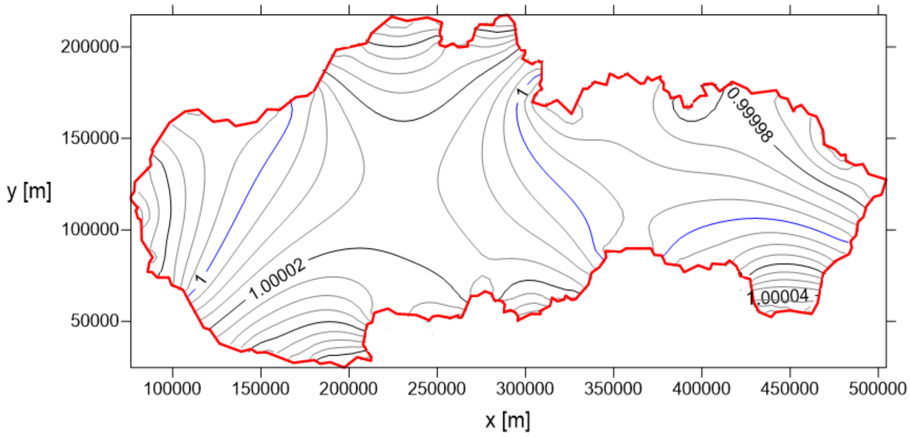


Fig. 8 Isometric lines of the scale distortion factor *m* for the Conformal conic projection with minimizing



RMS in an oblique aspect

Fig. 9 Isometric lines of the scale distortion factor *m* for the Conformal conic projection with minimizing RMS in an oblique aspect using LE solved by FEM

The percentage representation of the values of scale distortion in the Slovak Republic according to the given intervals (range 2 cm/km) in cartographic projection types **C)**, **D)** and their modification using LE solved by FEM is shown in the graph (Fig. 10). The scale distortions in the Conformal conic projection in an oblique aspect are indicated by a dark blue column, which shows that almost 45% of the area has a scale distortion in the range from –4 cm/km to +4 cm/km. The Conformal conic projection with minimizing RMS in an oblique aspect, which is shown in dark green, shows that up to 96% of the displayed territory has scale distortion values from the equal interval, which is twice as much as in the previous projection. We can see, that the modified Conformal conic projection with minimizing RMS in an oblique aspect has smaller distortion over a larger area. In the graph (Fig. 10), modifications of mentioned projections from the numerical method for solving LE using FEM are expressed by light blue, resp. light green colour. Based on these results we can see that projections by numerical method for solving LE by FEM have a better distribution of scale distortions throughout the territory.

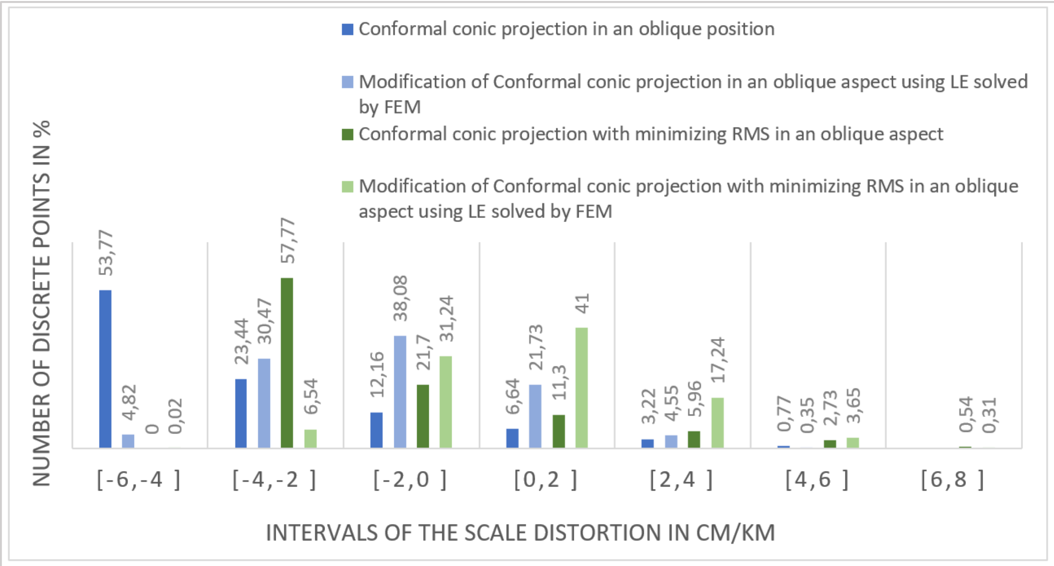


Fig. 10 The percentage of the scale distortion values in the territory of Slovakia according to the intervals in projection types **C)** and **D)** and their modifications using LE solved by FEM

The following figures (Fig. 11 – 14) show the graphic representation of the differences between the scale distortion values in the discrete points in the proposed cartographic projection types **A)**, **B)**, **C)**, **D)** and their modifications by the numerical method for solving LE using FEM. The differences in scale distortion values between analysed mentioned cartographic projections **A)**, **B)**, **C)**, **D)** and their modifications using LE solved by the numerical method FEM and also their standard deviations are listed in the following Tab. 3. The isometric line with zero difference in all images is marked in blue.

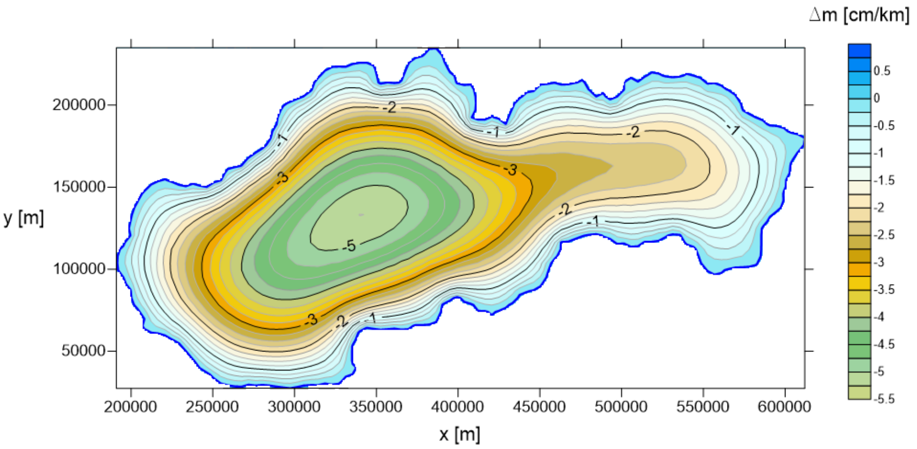


Fig. 11 Differences of the scale distortion values between Lambert´s conformal conic projection SR and its modification using LE solved by FEM

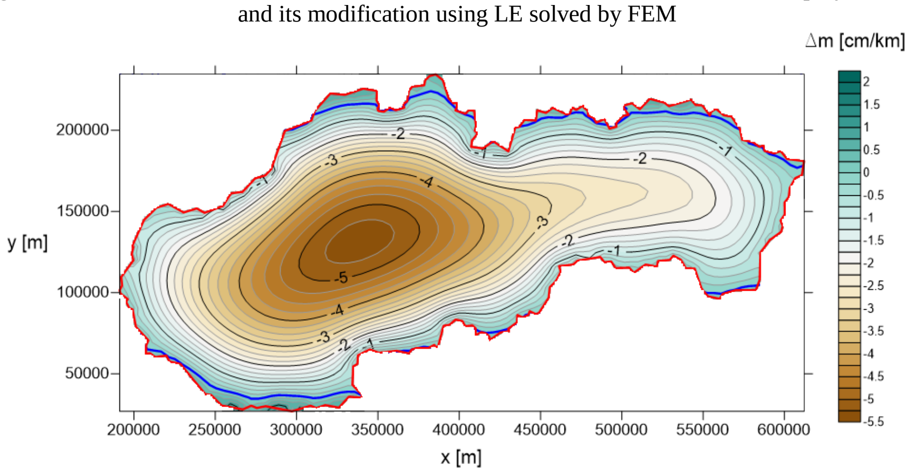


Fig. 12 Differences of the scale distortion values between Conformal conic projection with minimizing RMS in a normal aspect and its modification using LE solved by FEM

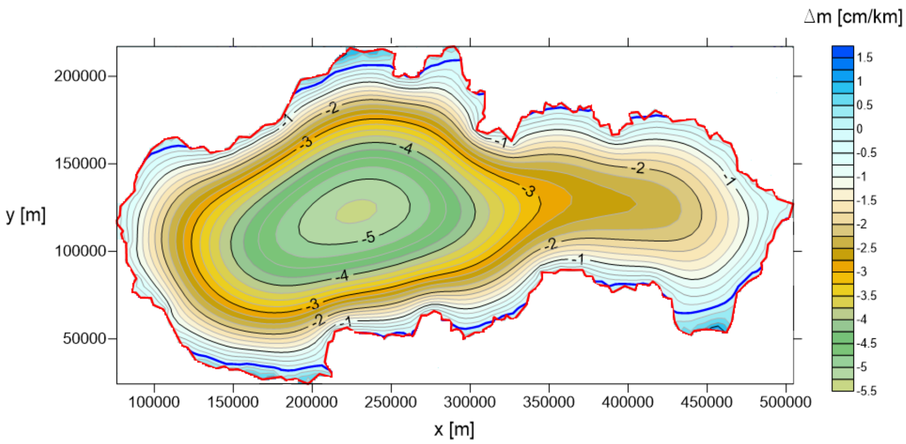


Fig. 13 Differences of the scale distortion values between Conformal conic projection in an oblique aspect and its modification using LE solved by FEM

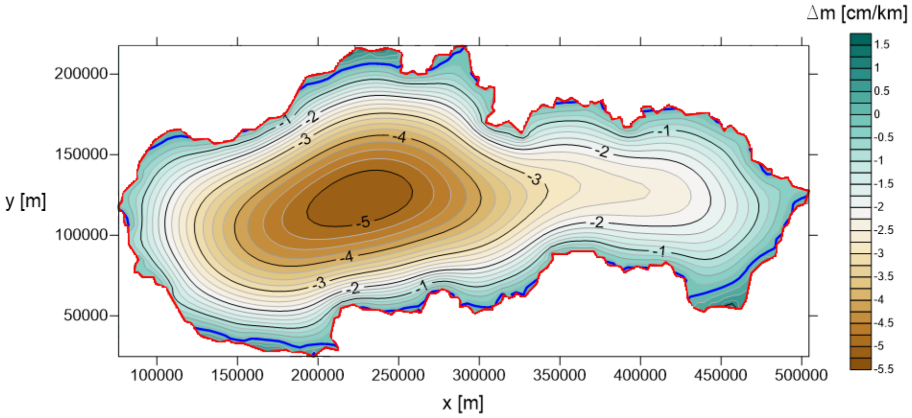


Fig. 14 Differences of the scale distortion values between Conformal conic projection with minimizing RMS in an oblique aspect and its modification using LE solved by FEM

Tab. 3 Differences of the scale distortion values in the Slovakia territory between the analysed cartographic projections A), B), C), D) and their modifications using the numerical method FEM for solving LE

Methods of calculation	Differences of scale distortion values		Standard deviation
	From	To	
Lambert's conformal conic projection and its modification using LE solved by FEM	-5.25 cm/km	0.0005 cm/km	1.53 cm/km
Conformal conic projection with minimizing RMS in a normal aspect and its modification using LE solved by FEM	-5.38 cm/km	1.56 cm/km	1.56 cm/km
Conformal conic projection in an oblique aspect and its modification using LE solved by FEM	-5.29 cm/km	1.49 cm/km	1.54 cm/km
Conformal conic projection with minimizing RMS in an oblique aspect and its modification using LE solved by FEM	-5.43 cm/km	1.43 cm/km	1.52 cm/km

Conclusions

The presented analysis showed a positive effect of using the Laplace equation solved by Finite Element Method on the scale distortion values in the above cartographic projections. In terms of maximum values of scale distortion (from –5.4 to 5.4 cm/km), a Conformal conic projection in the oblique aspect proposed in (Vajsáblová, 2015) proved to be the most effective for Slovakia territory. The position of the conical surfaces is different from that in Křovák's projection, which is currently used in Slovakia. Conformal conic projection optimized by minimizing RMS of scale distortion values in an oblique aspect has a more efficient distribution of scale distortion over the area (small distortion over a larger area), as shown by the value 2.7 cm/km of Airy-Kavraiskii's variational criterion. The value of this criterion for modification of this cartographic projection using LE solved by FEM is 1.9 cm/km. However, a disadvantage of conical projections in an oblique aspect is the need to transform the ellipsoid into a sphere. From this point of view, for the requirement of the Geodesy, Cartography and Cadastre Authority of the Slovak Republic, it is more effective proposal Lambert conformal conical projection (Vajsáblová, 2011), which is in a normal aspect and directly transforms the points of the ellipsoid GRS80 into a cone and the values of scale distortion are from –6.7 to 6.7 cm/km. From the point of view of the distribution of scale distortion in the area of the territory of Slovakia, a suitable alternative is also the Conformal conic projection optimized by minimizing RMS of scale distortion values in a normal aspect (Vajsáblová, 2015), as shown by the value 3.4 cm/km of Airy-Kavraiskii's variational criterion. The value of this criterion for modification of this cartographic projection using LE solved by FEM is 2.4 cm/km.

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R e s u m é

Modifikácia skreslení v kužeľovom kartografickom zobrazení Slovenska pomocou Laplaceovej rovnice riešenej metódou konečných prvkov

Mnohé matematické modely fyzikálnych problémov, s ktorými sa stretávame v inžinierskych odvetviach, podobne ako v optimalizácii skreslení v kartografickom zobrazení, sú formulované v tvare diferenciálnych rovníc. Jednou z možností ako riešiť takéto úlohy je použiť numerické metódy. Metóda konečných prvkov (MKP) je jednou z najznámejších a najrozšírenejších numerických metód. Medzi typické oblasti využitia MKP patrí simulácia prúdenia tepla, modelovanie gravitačného poľa Zeme alebo riešenie rôznych geodetických okrajových úloh.

Hodnoty skreslenia zobrazených prvkov na mapách závisia od prístupu k tvorbe kartografického zobrazenia. Hlavnými aspektmi v tvorbe kartografického zobrazenia vhodného pre dané územie je charakteristika geometrických vlastností územia, takisto matematické nástroje na výpočet vhodných parametrov s cieľom minimalizácie hodnôt skreslení. Tvar Slovenskej republiky (SR) po zániku bývalej Československej republiky nezodpovedá v súčasnosti používanému a platnému konformnému kužeľovému zobrazeniu vo všeobecnej polohe, ktoré bolo navrhnuté Ing. Josefom Křovákom. V roku 2010 bolo na základe požiadavky Úradu geodézie kartografie a katastra SR vypracované Lambertovo konformné kužeľové zobrazenie s parametrami pre SR (Vajsáblová, 2011).

Hlavným cieľom tohto článku je predstaviť rôzne typy kužeľových zobrazení vhodných pre územie SR, a to v normálnej a vo všeobecnej polohe (Vajsáblová, 2015) a ich modifikáciu pomocou Laplaceovej rovnice, ktorá je riešená pomocou numerickej metódy, konkrétne metódou konečných prvkov. Parametre týchto kužeľových zobrazení sú určené z kritérií na skreslenie rovnobežkových kružníc na projektovanom území, ako aj optimalizáciou dĺžkového skreslenia metódou minimalizácie strednej kvadratickej hodnoty dĺžkového skreslenia na zobrazovanom území použitím Airyho-Kavrajského kritéria.

V príspevku sú ukázané modifikácie hodnôt skreslenia v kartografických zobrazeniach riešením Laplaceovej rovnice riešenej pomocou MKP s okrajovou podmienkou (OP) na základe rovinných súradníc *x, y* a hodnôt modulu *m* dĺžkového skreslenia na území Slovenska daného jeho hranicou v nasledujúcich štyroch konformných kužeľových zobrazeniach:

A) Lambertovo konformné kužeľové zobrazenie pre SR v normálnej polohe,
B) Konformné kužeľové zobrazenie s minimalizáciou strednej kvadratickej hodnoty dĺžkového skreslenia v normálnej polohe,
C) Konformné kužeľové zobrazenie vo všeobecnej polohe,
D) Konformné kužeľové zobrazenie s minimalizáciou strednej kvadratickej hodnoty dĺžkového skreslenia vo všeobecnej polohe.

Podrobný popis a parametre návrhov zobrazení A – D sú uvedené v kapitole 1.2. V článku sa nachádzajú hodnotenia navrhovaných konformných zobrazení na základe dosiahnutých dĺžkových skreslení, a to extrémnych a hodnotenie podľa Airyho-Kavrajského variačného kritéria (1). Vykonaná analýza obsahuje ich vzájomné porovnanie, a tiež porovnanie s modifikovanými zobrazeniami, ktoré boli vytvorené z riešenia Laplaceovej rovnice pomocou numerickej metódy MKP (Tab. 1, Tab.2, Tab. 3).

Po porovnaní navrhnutých zobrazení na základe dosiahnutých extrémnych skreslení a optimálneho rozloženia skreslenia na ploche územia Slovenskej republiky konštatujeme nasledovné závery:

– Z hľadiska extrémnych dĺžkových skreslení má najefektívnejšie dĺžkové skreslenia (±5,4 cm/km) konformné kužeľové zobrazenie vo všeobecnej polohe.
– Z hľadiska štatistického rozloženia dĺžkového skreslenia na ploche územia sa najvýhodnejšie zobrazenie ukazuje konformné kužeľové zobrazenie s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia vo všeobecnej polohe, pričom izometrické čiary sú v obraze kartografických rovnobežkových kružníc (Obr. 10).
– Modifikované zobrazenia na základe riešenia Laplaceovej rovnice pomocou MKP majú minimalizovanú hodnotu dĺžkových skreslení na ploche územia pomocou Airyho-Kavrajského variačného kritéria (3,2 a 2,4 cm/km, resp. 2,1 a 1,9 cm/km).
– Nevýhodou zobrazení vo všeobecnej polohe je nutnosť transformácie elipsoidu na sféru, a teda návrh Lambertovho zobrazenia pre SR (Vajsáblová, 2011), príp. Konformné kužeľové zobrazenie s minimalizáciou RMS dĺžkového skreslenia na ploche územia SR v normálnej polohe (Vajsáblová, 2015), ktoré priamo transformujú body elipsoidu GRS80 na kužeľovú plochu, sú efektívne pre účely Úradu geodézie kartografie a katastra SR.

Obr. 1 Izometrické čiary modulu *m* dĺžkového skreslenia v Lambertovom konformnom kužeľovom zobrazení v normálnej polohe

Obr. 2 Izometrické čiary modulu *m* dĺžkového skreslenia pre modifikáciu Lambertovho konformného kužeľového zobrazenia v normálnej polohe na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 3 Izometrické čiary modulu *m* dĺžkového skreslenia konformného kužeľového zobrazenia s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia v normálnej polohe

Obr. 4 Izometrické čiary modulu *m* dĺžkového skreslenia pre modifikáciu konformného kužeľového zobrazenia s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia v normálnej polohe na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 5 Percentuálne rozloženie hodnôt dĺžkových skreslení na území Slovenska podľa intervalov v kartografických zobrazeniach typu A, B a ich modifikáciách na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 6 Izometrické čiary modulu *m* dĺžkového skreslenia v konformnom kužeľovom zobrazení vo všeobecnej polohe

Obr. 7 Izometrické čiary modulu *m* dĺžkového skreslenia pre modifikáciu konformného kužeľového zobrazenia vo všeobecnej polohe na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 8 Izometrické čiary modulu *m* dĺžkového skreslenia konformného kužeľového zobrazenia s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia vo všeobecnej polohe

Obr. 9 Izometrické čiary modulu *m* dĺžkového skreslenia pre modifikáciu konformného kužeľového zobrazenia s minimálnou strednou kvadratickou hodnotou dĺžkového skreslenia vo všeobecnej polohe na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 10 Percentuálne rozloženie hodnôt dĺžkových skreslení na území Slovenska podľa intervalov v kartografických zobrazeniach typu C, D a ich modifikáciách na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 11 Rozdiely hodnôt dĺžkových skreslení medzi Lambertovým konformným kužeľovým zobrazením a jeho modifikáciou na základe Laplaceovej rovnice riešenej pomocou MKP

Obr. 12 Rozdiely hodnôt dĺžkových skreslení medzi konformným kužeľovým zobrazením s minimálnou

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